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THE NUMBER OF MINIMAL LATTICE PATHS RESTRICTED BY TWO PARALLEL LINES

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We deal with non-decreasing paths on the non-negative quadrant of the integral square lattice, called by minimal lattice paths, from $(0,0)$ to a point (n,m) restricted by two parallel lines with an incline k (≥ 0). We express the generating functions of the number of these distinct minimal lattice paths in terms of the polynomials

$$\varphi_k(n,x)=\sum_{l=0}^{[n/k]}\binom{n-kl}{l}(-x)^l,\quad n\geq 0.$$

Formulas obtained thus include the generating function of the so-called higher Catalan number $C_k(n)$ or Ballot numbers as the special case.

The number of minimal lattice paths for $k=1$ is given as an explicit form by expanding the corresponding generating function.

1. Introduction

Consider a particle executing a random walk on the interval $[-b,a]$ where a and b are positive integers. The particle in a point j $(-b<j<a)$ moves to one of the points $j-1$ and $j+1$ in a single transition. Both of the points a and $-b$ are taken to be absorbing barriers. First, we consider random walk paths in which the particle arrives at a point i $(-b<i<a)$ after t transitions, starting from an initial point 0 without touching any of the absorbing barriers a and $-b$. If the paths consist of n-right moves and m-left moves, then each of these paths corresponds to a non-decreasing path on the nonnegative quadrant of the integral square lattice, called minimal lattice path hereafter, from $(0,0)$ to the point (n,m), without crossing any of the lines $y=x-a+1$ and $y=x+b-1$, where $n+m=t$ and $n-m=i$. The number of distinct paths satisfying these restrictions is given by (S. Dua et al. [1])

$$\sum_{l=-\infty}^{\infty}\left\{\binom{t}{\frac{1}{2}(t+i)+l(a+b)}-\binom{t}{\frac{1}{2}(t+i)+b+l(a+b)}\right\},\quad t+i\equiv 0 \ (\mathrm{mod}\ 2)\quad (1)$$

with the usual convention that $\binom{t}{n}=0$ for $n>t$ or $n<0$.

In this present paper, we deal with generating functions of the following numbers $\{W_k(N/k, n, r), n \geq 0\}$, and $\{T_k(N/k, n, r, s), n \geq 0\}$, respectively:

(i) $W_k(N/k, n, r)$ is the number of minimal lattice paths from $(0, 0)$ to $(n, kn - N + r)$ without crossing a line $y = k(x - N/k)$ (see Fig. 1),

(ii) $T_k(N/k, n, r, s)$ is the number of minimal lattice paths from $(0, 0)$ to $(n, kn - N + r)$ without crossing any of lines $y = k(x - N/k)$ and $y = kx + s$ (see Fig. 2), where $n, N, r, s \geq 0$ and $k \geq 1$.

According to the above notations, the finite sum (1) is equal to $T_1(a-1, n, a-1-i, b-1)$, where $n = \frac{1}{2}(t+i)$ and $t + i \equiv 0 \pmod 2$.

The number $W_k(0, n, 0)$ of minimal lattice paths from $(0, 0)$ to (n, kn) without crossing the line $y = kx$ is the well known higher Catalan number (Rogers [2])

$$C_k(n) = \frac{1}{kn+1}\binom{(k+1)n}{n}, \quad n \geq 0.$$

The higher Catalan number $C_k(n)$, like the Catalan number $C_1(n)$, occurs in a wide variety of combinatorial problems [2–5]. The number $W_k(0, n, r)$ of minimal lattice paths from $(0, 0)$ to $(n, kn + r)$ without crossing the line $y = kx$ is also

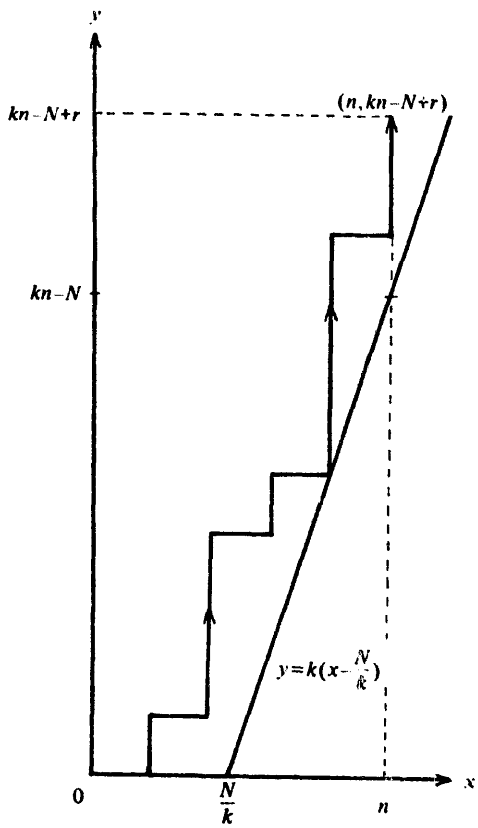

Fig. 1. $W_k(N/k, n, r)$.

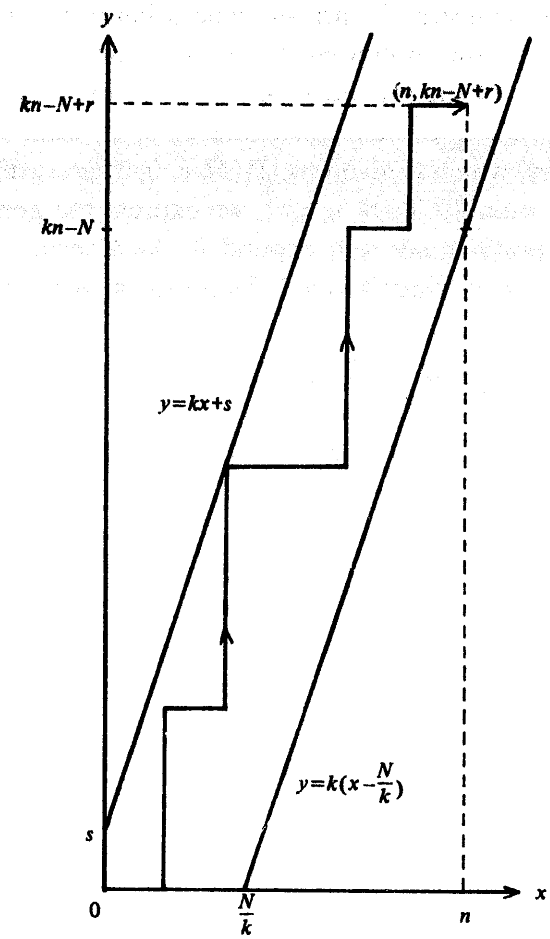

Fig. 2. $T_k(N/k, n, r, s)$.

represented by (Rogers [2])

$$W_k(0, n, r) = A_n(r+1, k+1) = \frac{r+1}{r+1+(k+1)n}\binom{r+1+(k+1)n}{n}, \quad n \geqslant 0$$

with $A_n(0, k+1) = C_k(n)$.

In our previous paper [6], we have expressed $W_k(N/k, n, r)$ in terms of $\{A_l(r+1, k+1)\}$ for $N, n, r \geqslant 0$ and $kn-N+r \geqslant 0$. In Section 2, we give the generating functions for $\{W_k(N/k, n, r), n \geqslant 0\}$. In Section 3, we express $T_k(N/k, n, r, s)$ in terms of $\{W_k(N'/k, l, r')\}$. After that, we obtain the generating functions for $\{T_k(N/k, n, r, s), n \geqslant 0\}$ where $N, s \geqslant 0$ and $0 \leqslant r \leqslant s+N$. It is proved that the generating function for $\{T_k(N/k, n, r, s), n \geqslant 0\}$ converges to the one for $\{W_k(N/k, n, r), n \geqslant 0\}$ as $s \to \infty$. As will be seen in these sections the polynomials

$$\varphi_k(n, x) = \sum_{l=0}^{[n/k]} \binom{n-kl}{l}(-x)^n, \quad n \geqslant 0 \tag{2}$$

take an important place in our theory.

In Section 4, we are concerned with the generating functions for the particular case $k=1$. For $k=1$, the polynomials $\varphi_1(n,x)$ are closely related to the Chebyshev polynomials of the second kind. C.J. Everett et al. [7] has sought the roots of $\varphi_1(n,x)$, in order to evaluate the number of random walk paths between two absorbing barriers, which is given by $T_1(N,n,0,0)$ according to our notation. So as to evaluate the number $T_1(N,n,r,s)$, we express the generating function in terms of Chebyshev polynomials and expand it. As a result, we find an explicit expression for $T_1(N,n,r,s)$ which has a different form from (1).

2. Generating function for $\{W_k(N/k,n,r), n\geqslant 0\}$

As a preparation for deriving the generating functions for $\{W_k(N/k,n,r), n\geqslant 0\}$, let us quote the following two lemmas:

Lemma 1 (Pólya and Szegö [8], Gould [9]). *If a function $u(x)$ satisfies the equation $u=1+xu^{\beta}$ for an integer β and $u(0)=1$, then the following expansions are valid for any integer α:*

$$u^{\alpha}=\sum_{n=0}^{\infty}A_n(\alpha,\beta)x^n,\quad \alpha\neq 0 \tag{3}$$

$$\frac{u^{\alpha+1}}{(1-\beta)u+\beta}=\sum_{n=0}^{\infty}\binom{\alpha+\beta n}{n}x^n \tag{4}$$

for $|x|<|(\beta-1)^{\beta-1}/\beta^{\beta}|$, where

$$A_n(\alpha,\beta)=\frac{\alpha}{\alpha+\beta n}\binom{\alpha+\beta n}{n},\quad n\geqslant 0 \tag{5}$$

Lemma 2 (Gould [9]). *For integers α_1, α_2 and β, the following convolution of Vandermonde's type is valid:*

$$\sum_{l=0}^{n}\binom{\alpha_1+\beta l}{l}A_{n-l}(\alpha_2,\beta)=\binom{\alpha_1+\alpha_2+\beta n}{n} \tag{6}$$

where $A_m(\alpha_2,\beta)$ is defined by (5) for $m=0,\ldots,n$.

In our previous paper [6], we have expressed $W_k(N/k,n,r)$ in terms of $\{A_l(r+1,k+1)\}$ defined by (5) with $\alpha=r+1$ and $\beta=k+1$ as follows:

Lemma 3 (T. Cong and Sato [6]). *For $N,n,r\geqslant 0$ and $k\geqslant 1$, $W_k(N/k,n,r)$ is given by*

$$W_k\left(\frac{N}{k},n,r\right)=\begin{cases}\dbinom{(k+1)n-N+r}{n}, & 0\leqslant n<\left[\dfrac{N}{k}\right], \quad (7)\\[2ex] \displaystyle\sum_{l=0}^{[N/k]}\binom{(k+1)l-N-1}{l}A_{n-l}(r+1,k+1), & n\geqslant\left[\dfrac{N}{k}\right]+1. \quad (8)\end{cases}$$

Note that if $N \geqslant r+1$, we define, for convenience, $W_k(N/k, n, r)$ by the right hand side of (7) for $0 \leqslant n \leqslant [(N-r-1)/k]$.

Defining the generating function

$$W_k\left(\frac{N}{k}, x, r\right) = \sum_{n=0}^{\infty} W_k\left(\frac{N}{k}, n, r\right)x^n, \quad N, r \geqslant 0,$$

then we have the following:

Theorem 1. *For $N, r \geqslant 0$, the generating functions of $W_k(N/k, n, r)$, $n \geqslant 0$, are given by*

$$W_k(N/k, x, r) = u^{r+1}\varphi_k(N, x) \tag{9}$$

where $\varphi_k(N, x)$ is defined by (2) and the function u is defined in Lemma 1 with $\beta = k+1$.

Proof. For $n = 0, \ldots, [N/k]$, the right hand side of (8) becomes, using the convolution (6),

$$\begin{aligned}\sum_{l=0}^{[N/k]} \binom{(k+1)l-N-1}{l} A_{n-l}(r+1, k+1) &= \sum_{l=0}^{n} \binom{(k+1)l-N-1}{l} A_{n-l}(r+1, k+1) \\ &= \binom{(k+1)n-N+r}{n}\end{aligned}$$

where $A_n(r+1, k+1) = 0$ for $n < 0$. Therefore equation (8) is valid for $n \geqslant 0$, so that we have

$$\begin{aligned}W_k\left(\frac{N}{k}, x, r\right) &= \sum_{n=0}^{\infty} \sum_{l=0}^{[N/k]} \binom{(k+1)l-N-1}{l} A_{n-l}(r+1, k+1)x^n \\ &= \sum_{l=0}^{[N/k]} \binom{(k+1)l-N-1}{l} \sum_{n=l}^{\infty} A_{n-l}(r+1, k+1)x^n \\ &= \sum_{l=0}^{[N/k]} \binom{(k+1)l-N-1}{l} x^l \sum_{n=0}^{\infty} A_n(r+1, k+1)x^n \\ &= \sum_{l=0}^{[N/k]} \binom{N-kl}{l} (-x)^l \sum_{n=0}^{\infty} A_n(r+1, k+1)x^n.\end{aligned}$$

Putting $\alpha = r+1$ and $\beta = k+1$ in the expansion (3), we find finally

$$W_k(N/k, x, r) = u^{r+1}\varphi_k(N, x),$$

where the function u satisfies the equation $u = 1 + xu^{k+1}$ and $\varphi_k(N, x)$ is defined by (2). $\square$

Corollary 1. *For $N \geqslant r+1$,*

$$\sum_{n=[(N-r-1)/k]+1}^{\infty} W_k\left(\frac{N}{k}, n, r\right)x^n = u^{r+1}\varphi_k(N, x) - \varphi_k(N-r-1, x) \tag{10}$$

where the function u satisfies the equation $u = 1 + xu^{k+1}$ *and* $\varphi_k(n, x)$ *is defined by* (2).

Proof. From (7), it follows immediately that

$$\sum_{n=0}^{[(N-r-1)/k]} W_k\left(\frac{N}{k}, n, r\right)x^n = \sum_{n=0}^{[(N-r-1)/k]} \binom{(k+1)n-N+r}{n}x^n = \varphi_k(N-r-1, x),$$

which, in combination with (9), completes the proof. □

3. Generating function for $\{T_k(N/k, n, r, s), n \geq 0\}$

Now consider a minimal lattice path restricted by two parallel straight lines with an incline k (≥ 1), with which we are particularly concerned in this paper.

$T_k(N/k, n, s+N, s)$ is the number of minimal paths from $(0, 0)$ to $(n, kn+s)$ without crossing any of the lines $y = k(x - N/k)$ and $y = kx + s$ for $N, n, s \geq 0$ (see Fig. 3). Let

$$T_k\left(\frac{N}{k}, x, s+N, s\right) = \sum_{n=0}^{\infty} T_k\left(\frac{N}{k}, n, s+N, s\right)x^n$$

be the generating function for $\{T_k(N/k, n, s+N, s), n \geq 0\}$.

Theorem 2. *For* $N, s \geq 0$,

$$T_k\left(\frac{N}{k}, x, s+N, s\right) = \frac{\varphi_k(N, x)}{\varphi_k(N+s+1, x)} \tag{11}$$

where $\varphi_k(n, x)$ *is defined by* (2).

Proof. For $n = 0, \ldots, [N/k]$, $T_k(N/k, n, s+N, s)$ is the number of minimal lattice paths from $(0, 0)$ to $(n, kn+s)$ without crossing the line $y = kx + s$. This is given by $W_k(0, n, s)$, that is,

$$T_k(N/k, n, s+N, s) = A_n(s+1, k+1) \quad \text{for } 0 \leq n \leq [N/k]. \tag{12}$$

While, for $n \geq [N/k]+1$, let us consider a minimal lattice path from $(0, 0)$ to $(n, kn+s)$ crossing the line $y = kx + s$ but not the line $y = k(x - N/k)$. $T_k(N/k, l, s+N, s)$ is the number of minimal lattice paths from $(0, 0)$ to $(l, kl+s)$ without crossing any of the lines $y = k(x - N/k)$ and $y = kx + s$, and $W_k((s+N+1)/k, n-l, s+N)$ is the number of minimal lattice paths from $(l, kl+s+1)$ to $(n, kn+s)$ without crossing the line $y = k(x - N/k)$ for $l = 0, 1, \ldots, n-1$. Therefore

$$\sum_{l=0}^{n-1} T_k\left(\frac{N}{k}, l, s+N, s\right) W_k\left(\frac{s+N+1}{k}, n-l, s+N\right)$$

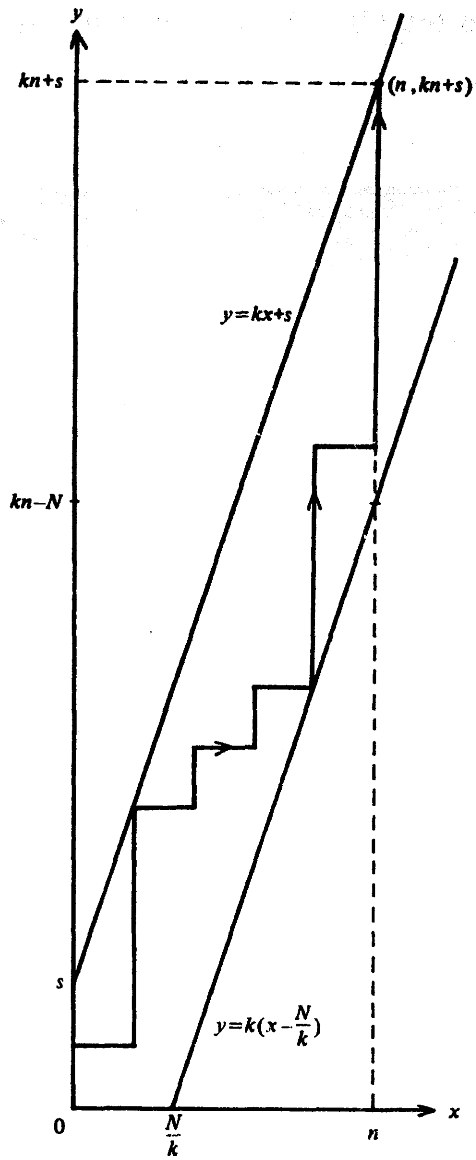

Fig. 3. $T_k(N/k, n, s+N, s)$.

is the number of minimal lattice paths from $(0,0)$ to $(n, kn+s)$ crossing the line $y=kx+s$ but not crossing the line $y=k(x-N/k)$. Hence we have the following recurrence relation for $T_k(N/k, n, s+N, s)$:

$$T_k\left(\frac{N}{k}, n, s+N, s\right) = W_k\left(\frac{N}{k}, n, s+N\right) - \sum_{l=0}^{n-1} T_k\left(\frac{N}{k}, l, s+N, s\right) W_k\left(\frac{s+N+1}{k}, n-l, s+N\right)$$

for $n \geq [N/k]+1$. Since $W_k((s+N+1)/k, 0, s+N)=1$ holds for $s, N \geq 0$, we have

$$\sum_{l=0}^{n} T_k\left(\frac{N}{k}, l, s+N, s\right) W_k\left(\frac{s+N+1}{k}, n-l, s+N\right) = W_k\left(\frac{N}{k}, n, s+N\right) \quad (13)$$

for $n \geq [N/k]+1$. From (6), (7) and (12), the above equation (13) is valid for $n \geq 0$, which leads to

$$\left(\sum_{n=0}^{\infty} T_k\left(\frac{N}{k}, n, s+N, s\right)x^n\right)\left(\sum_{n=0}^{\infty} W_k\left(\frac{s+N+1}{k}, n, s+N\right)x^n\right) = \sum_{n=0}^{\infty} W_k\left(\frac{N}{k}, n, s+N\right)x^n.$$

By appealing to Theorem 1, we have

$$T_k\left(\frac{N}{k}, x, s+N, s\right) = \frac{u^{s+N+1}\varphi_k(N, x)}{u^{s+N+1}\varphi_k(s+N+1, x)} = \frac{\varphi_k(N, x)}{\varphi_k(s+N+1, x)}. \quad \square$$

Let us define the generating functions

$$T_k\left(\frac{N}{k}, x, r, s\right) = \sum_{n=0}^{\infty} T_k\left(\frac{N}{k}, n, r, s\right)x^n$$

for $N, s \geq 0$ and $0 \leq r \leq s+N$. Note that if $N \geq r+1$, we define, for convenience,

$$T_k(N/k, n, r, s) = W_k(N/k, n, r) \quad \text{for } 0 \leq n \leq [(N-r-1)/k]. \tag{14}$$

Theorem 3. *For $N, s \geq 0$ and $0 \leq r \leq s+N$, we have*

$$T_k\left(\frac{N}{k}, x, r, s\right) = \frac{\varphi_k(N, x)\varphi_k(s+N-r, x)}{\varphi_k(s+N+1, x)} \tag{15}$$

where $\varphi_k(n, x)$ is defined by (2).

Proof. For $n = \max(0, [(N-r-1)/k]+1), \ldots, [(s+N-r)/k]-1, [(s+N-r)/k]$, $T_k(N/k, n, r, s)$ is the number of minimal lattice paths from $(0, 0)$ to $(n, kn-N+r)$ without crossing the line $y = k(x-N/k)$, which is given by $W_k(N/k, n, r)$. If $N \geq r+1$, (14) is valid for $0 \leq n \leq [(N-r-1)/k]$ as mentioned above. Hence, we have

$$T_k(N/k, n, r, s) = W_k(N/k, n, r), \quad 0 \leq n \leq n_0 - 1, \tag{16}$$

where $[(s+N-r)/k] = n_0 - 1$.

While, for $n \geq n_0$, let us consider a minimal lattice path from $(0, 0)$ to $(n, kn-N+r)$ crossing the line $y = kx+s$ but not the line $y = k(x-N/k)$. $T_k(N/k, l, s+N, s)$ is the number of minimal lattice paths from $(0, 0)$ to $(l, kl+s)$ without crossing any of the lines $y = k(x-N/k)$ and $y = kx+s$, and $W_k((s+N+1)/k, n-l, r)$ is the number of minimal lattice paths from $(l, kl+s+1)$ to $(n, kn-N+r)$ without crossing the line $y = k(x-N/k)$ for $l = 0, \ldots, n-n_0$.

Consequently, we have

$$T_k\left(\frac{N}{k}, n, r, s\right) = W_k\left(\frac{N}{k}, n, r\right) - \sum_{l=0}^{n-n_0} T_k\left(\frac{N}{k}, l, s+N, s\right) W_k\left(\frac{s+N+1}{k}, n-l, r\right) \tag{17}$$

for $n \geqslant n_0$. From (16) and (17), it follows that

$$\begin{aligned}
T_k\left(\frac{N}{k}, x, r, s\right) &= \sum_{n=0}^{\infty} W_k\left(\frac{N}{k}, n, r\right)x^n \\
&\quad - \sum_{n=n_0}^{\infty} \sum_{l=0}^{n-n_0} T_k\left(\frac{N}{k}, l, s+N, s\right) W_k\left(\frac{s+N+1}{k}, n-l, r\right)x^n \\
&= \sum_{n=0}^{\infty} W_k\left(\frac{N}{k}, n, r\right)x^n \\
&\quad - \sum_{l=0}^{\infty} T_k\left(\frac{N}{k}, l, s+N, s\right) \sum_{n=n_0+l}^{\infty} W_k\left(\frac{s+N+1}{k}, n-l, r\right)x^n \\
&= \sum_{n=0}^{\infty} W_k\left(\frac{N}{k}, n, r\right)x^n - \left(\sum_{l=0}^{\infty} T_k\left(\frac{N}{k}, l, s+N, s\right)x^l\right) \\
&\quad \times \left(\sum_{n=n_0}^{\infty} W_k\left(\frac{s+N+1}{k}, n, r\right)x^n\right).
\end{aligned}$$

Applying Theorems 1, 2 and Corollary 1, we finally reach

$$\begin{aligned}
T_k\left(\frac{N}{k}, x, r, s\right) &= u^{r+1}\varphi_k(N, x) \\
&\quad - \frac{\varphi_k(N, x)}{\varphi_k(s+N+1, x)}\{u^{r+1}\varphi_k(s+N+1, x) - \varphi_k(N+s-r, x)\} \\
&= \frac{\varphi_k(N, x)\varphi_k(s+N-r, x)}{\varphi_k(s+N+1, x)}
\end{aligned}$$

where the function u is defined by Theorem 1. □

As the right hand side of (15) is a ratio of two polynomials in x, explicit expression for $T_k(N/k, n, r, s)$ can be obtained by expanding it. For instance, $\varphi_k(n, x)^{-1}$ can be expanded in the following form:

$$\frac{1}{\varphi_k(n, x)} = \sum_{m=0}^{\infty} \left[\sum_{\substack{l_1+\cdots+l_j=m \\ 1 \leq l_i \leq [n/k], i=1,\ldots,j}} (-1)^{m+j} \binom{n-kl_1}{l_1} \cdots \binom{n-kl_j}{l_j} \right] x^m \tag{18}$$

Since the generating function (15)for $N=0$ and $r=s$ reduces to

$$T_k(0, x, s, s) = \frac{1}{\varphi_k(s+1, x)},$$

and together with (18), the explicit expression of $T_k(0, n, s, s)$ is given by

$$T_k(0, n, s, s) = \sum_{\substack{l_1+\cdots+l_j=n \\ 1\le l_i\le[(s+1)/k], i=1,\ldots,j}} (-1)^{n+j}\binom{s+1-kl_1}{l_1}\cdots\binom{s+1-kl_j}{l_j}.$$

Now in order to show that the generating function $T_k(N/k, x, r, s)$ converges to $W_k(N/k, x, r)$ as $s \to \infty$, we notice:

Lemma 3. *For $k\ge 1$ and $|x|<k^k/(k+1)^{k+1}$*

$$\lim_{n\to\infty} \varphi_k(n, x)u^{n+1} = \frac{u}{(k+1)-ku}, \tag{19}$$

where the function u is defined by Theorem 1.

Proof. Let a function u satisfy the equation $u=1+xu^\beta$ with $\beta=k+1$ and $u(0)=1$. From Lemma 1, we have

$$u^{n+1} = \sum_{l=0}^{\infty} A_l(n+1, \beta)x^l, \qquad \frac{u^{-n}}{(1-\beta)u+\beta} = \sum_{l=0}^{\infty}\binom{\beta l-n-1}{l}x^l$$

for $n\ge 0$. From these relations, it follows that

$$\begin{aligned}
&\frac{u}{(1-\beta)u+\beta}\\
&=\frac{u^{-n}}{(1-\beta)u+\beta}u^{n+1} = \sum_{l=0}^{[n/k]}\binom{\beta l-n-1}{l}x^l u^{n+1} + \sum_{l=[n/k]+1}^{\infty}\binom{\beta l-n-1}{l}x^l u^{n+1}\\
&=\sum_{l=0}^{[n/k]}\binom{\beta l-n-1}{l}x^l u^{n+1} + \sum_{l=[n/k]+1}^{\infty}\left(\sum_{m=[n/k]+1}^{l}\binom{\beta m-n-1}{m}A_{l-m}(n+1, \beta)\right)x^l.
\end{aligned} \tag{20}$$

Applying the convolution (6), the inequality

$$\begin{aligned}
&\sum_{m=[n/k]+1}^{l}\binom{\beta m-n-1}{m}A_{l-m}(n+1, \beta)\\
&\qquad< \sum_{m=0}^{l}\binom{\beta m-n-1}{m}A_{l-m}(n+1, \beta) = \binom{\beta l}{l}, \quad n\ge 0
\end{aligned}$$

follows for $l\ge[n/k]+1$. Since

$$\lim_{n\to\infty}\sum_{l=[n/k]+1}^{\infty}\binom{\beta l}{l}x^l = 0$$

holds for $|x|<|(\beta-1)^{\beta-1}/\beta^\beta|$, the second term of the right hand side of (20) converges to 0. Therefore we have

$$\frac{u}{(1-\beta)u+\beta} = \lim_{n\to\infty}\left[\sum_{l=0}^{[n/k]}\binom{\beta l-n-1}{l}x^l\right]u^{n+1}.$$

Putting $\beta = k+1$, tha above equation gives (19). $\square$

After this preparation, we have

Theorem 4. *For N, $r \geqslant 0$ and $k \geqslant 1$,*

$$\lim_{s\to\infty} T_k\left(\frac{N}{k}, x, r, s\right) = W_k\left(\frac{N}{k}, x, r\right), \quad |x| < \frac{k^k}{(k+1)^{k+1}}.$$

Proof. From Theorems 1, 2 and Lemma 3, we have

$$\begin{aligned}\lim_{s\to\infty} T_k\left(\frac{N}{k}, x, r, s\right) &= \lim_{s\to\infty} \frac{\varphi_k(N, x)\varphi_k(s+N-r, x)}{\varphi_k(s+N+1, x)} \\ &= \lim_{s\to\infty} \frac{u^{s+N-r+1}\varphi_k(s+N-r, x)}{u^{s+N+2}\varphi_k(s+N+1, x)} \varphi_k(N, x)u^{r+1} \\ &= u^{r+1}\varphi_k(N, x) = W_k\left(\frac{N}{k}, x, r\right). \qquad \square\end{aligned}$$

Next, we show that there is a difference equation which the polynomial $\varphi_k(n, x)$ satisfies, in the following:

Theorem 5. *The polynomial $\varphi_k(n, x)$ satisfies the difference equation*

$$\varphi_k(n, x) = \begin{cases} \varphi_k(n-1, x) - x\varphi_k(n-k-1, x), & n \geqslant k+1, \\ 1, & 0 \leqslant n \leqslant k. \end{cases} \tag{21}$$

Proof. This is clear since

$$\sum_{n=0}^{\infty} \varphi_k(n, x)z^n = 1/(1-z+xz^{k+1}). \qquad \square$$

4. Explicit expression for $T_1(N, n, r, s)$

In this section, we consider the particular case $k=1$. In this case, $W_1(N, n, r)$ is the number of minimal lattice paths from $(0, 0)$ to $(n, n-N+r)$ without crossing the line $y = x - N$ for $n, N, r \geqslant 0$ and $n \geqslant N-r$, which is simply given by

$$W_1(N, n, r) = \binom{2n-N+r}{n} - \binom{2n-N+r}{n-N-1}, \quad n \geqslant N-r$$

by means of the reflection principle. $T_1(N, n, r, s)$ is the number of minimal lattice paths from $(0, 0)$ to $(n, n-N+r)$ without crossing any of the lines $y = x - N$ and $y = x + s$ for $n, N, s \geqslant 0$, $0 \leqslant r \leqslant s+N$ and $n \geqslant N-r$. In order to evaluate $T_1(N, n, r, s)$, we express the generating functions in terms of Chebyshev polynomials.

Theorem 6. (i) *For* $N, r \geqslant 0$,

$$W_1(N, x, r) = \left(\frac{1-\sqrt{1-4x}}{2x}\right)^{r+1} x^{N/2} U_N\left(\frac{1}{2\sqrt{x}}\right).$$

(ii) *For* $N, s \geqslant 0$ *and* $0 \leqslant r \leqslant s+N$,

$$T_1(N, x, r, s) = x^{(N-r-1)/2} \frac{U_N(1/2\sqrt{x})U_{s+N-r}(1/2\sqrt{x})}{U_{s+N+1}(1/2\sqrt{x})} \tag{22}$$

where $U_n(x)$ *is Chebyshev polynomial of the second kind*

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta.$$

Proof. (i) and (ii) are trivial since the polynomial $\varphi_1(n, x)$ can be written as:

$$\varphi_1(n, x) = \sum_{l=0}^{[n/2]} \binom{n-l}{l}(-x)^l = x^{n/2} U_n\left(\frac{1}{2\sqrt{x}}\right), \quad n \geqslant 0. \qquad \square$$

In virtue of Theorem 6(ii), explicit expression for $T_1(N, n, r, s)$ can be obtained as follows:

Theorem 7. *For* $N, n, s \geqslant 0$ *and* $0 \leqslant r \leqslant s+N$,

$$T_1(N, n, r, s) = \frac{4}{s+N+2} \sum_{\nu=1}^{M} (2\cos\theta_\nu)^{2N+N} \sin(N+1)\theta_\nu \sin(r+1)\theta_\nu \tag{23}$$

where $\theta_\nu = \nu\pi/(s+N+2)$, $\nu = 1, \ldots, [\frac{1}{2}(s+N+1)]$ $(=M)$.

Proof. Let us put

$$x = (2\cos\theta)^{-2}, \quad 0 < \theta < \tfrac{1}{2}\pi.$$

From (22), it follows that

$$T_1(N, x, r, s) = (2\cos\theta)^{-(N-r-1)} \frac{\sin(N+1)\theta \sin(s+N-r+1)\theta}{\sin\theta \sin(s+N+2)\theta}.$$

There are $[\frac{1}{2}(s+N+1)]$ different roots of $\sin(s+N+2)\theta$:

$$\theta_\nu = \frac{\nu\pi}{s+N+2}, \quad \nu = 1, \ldots, [\tfrac{1}{2}(s+N+1)] \ (=M)$$

which imply M different roots of $\varphi_1(s+N+1, x)$ such that

$$x_\nu = (2\cos\theta_\nu)^{-2}, \quad \nu = 1, \ldots, M.$$

Consequently, the coefficient of x^n in the expansion of $T_1(N, x, r, s)$ is represented by

$$\sum_{\nu=1}^{M}(2\cos\theta_\nu)^{2n-N+r+3}\frac{\sin(N+1)\theta_\nu\,\sin(s+N-r+1)\theta_\nu}{(-\sin\theta_\nu)(s+N+2)(\cos(s+N+2)\theta_\nu)[d\theta/dx]_{\theta=\theta_\nu}}$$

$$=\frac{4}{s+N+2}\sum_{\nu=1}^{M}(2\cos\theta_\nu)^{2n+r-N}\sin(N+1)\theta_\nu\,\sin(r+1)\theta_\nu.$$

which give $T_1(N,n,r,s)$. □

The finite sum (1) as mentioned in Section 1 is equal to $T_1(a-1,n,a-1-i,b-1)$, where $n=\frac{1}{2}(t+i)$. From (23), it is immediately shown that

Corollary 2. *For $t+i\equiv 0 \pmod 2$ and $-b<i<a$,*

$$\sum_{l=-\infty}^{\infty}\left\{\binom{t}{\frac{1}{2}(t+i)+l(a+b)}-\binom{t}{\frac{1}{2}(t+i)+b+l(a+b)}\right\}$$

$$=\frac{4}{a+b}\sum_{\nu=1}^{M}(2\cos\theta_\nu)^t\sin a\theta_\nu\,\sin(a-i)\theta_\nu$$

where $\theta_\nu=\nu\pi/(a+b)$, $\nu=1,2,\ldots,[\frac{1}{2}(a+b-1)]$ $(=M)$.

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